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Properties and eigenvalues of a fifth label generating operator for quadrupole-phonon states

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Abstract. The properties of a previously algebraically derived fifth label generating operator S for quadrupole-phonon states are discussed. This algebraic operator is connected with the operators following from pure group theoretical principles. A method is developed by which it is possible to calculate numerically the eigenvalues of the operator S.

1. Introduction

The wavefunctions of the quadrupole-phonon states, as introduced by Bohr (1952), can be exactly defined by using group theoretical (Arima and Iachello 1976, Chacon *et al* 1976, Kemmer *et al* 1968, Von Bernus *et al* 1975, Corrigan *et al* 1976, Weber *et al* 1966, Williams and Pursey 1968) and pure algebraic techniques (Vanden Berghe and De Meyer 1979b). Five quantum numbers are needed to classify uniquely the states built up by N quadrupole phonon states. Four of them are related to the Casimir operators of the groups appearing in the chain $U(5)\supset R(5)\supset R(3)\supset R(2)$, i.e. the boson number N, the seniority v, the angular momentum J_N and its projection M_N. The fifth label which one usually introduces counts the number of boson triplets coupled to zero angular momentum (Arima and Iachello 1976, Chacon *et al* 1976) and is not related to the eigenvalue of an operator. On the contrary, it is known that there exists an integrity basis which gives all $R(5)\supset R(3)$ labelling operators (Gaskell *et al* 1978), the number of which is stated by a general theorem (Peccia and Sharp 1976). To our knowledge, however, no attempts have been made to calculate eigenvalues and to determine an orthonormalised basis.

Very recently a method has been developed by which it was possible to derive the most general form of operators which commute with the Casimir operators of the four mentioned groups and which are independent of them (De Meyer and Vanden Berghe 1980). The eigenvalue of such an operator can then be used as a fifth label. These operators have been developed in terms of a set of scalar operators, also called canonical operators, in order to refer to the property that all the phonon creation operators b^+_μ stand to the left of the phonon annihilation operators $(-1)^\mu b_{-\mu}$, i.e.

$$\begin{aligned} O^{I_2I_3\ldots I_{N-1}J_N}_{J_2J_3\ldots J_{N-1}J_N} &= \sqrt{2J+1}[((\ldots(b^+b^+)^{J_2}b^+)^{J_3}\ldots)^{J_{N-1}}b^+)^{J_N} \\ &\quad\times((\ldots(bb)^{I_2}b)^{I_3}\ldots)^{I_{N-1}}b)^{J_N}]^0 \qquad (N\geqslant 2) \\ O &= \sqrt{5}(b^+b)^0 \qquad (N=1). \end{aligned} \tag{1.1}$$

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It has been remarked that the set of operators (1.1) is overcomplete, since for fixed N and J_N values the operators having different intermediate angular momentum values are not necessarily independent. It has been shown (De Meyer and Vanden Berghe 1980) that it is possible to indicate for any fixed N and J_N, insofar as an N-phonon state with angular momentum J_N exists, a minimal set of independent operators $O_{J_2 \ldots J_N}^{I_2 \ldots J_N}$, in terms of which all other canonical operators can be expressed. The criterion for selecting this set of independent operators has been discussed in detail by Vanden Berghe and De Meyer (1979b). For further use we state here the expressions in canonical form of the number operator N and its square, the angular momentum operator J^2 and the Casimir operator of the R(5) group:

$$N = O \tag{1.2}$$

$$N^2 = O + \sum_J O_J^J \tag{1.3}$$

$$J^2 = 6O + 30 \sum_J \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & 1 \end{Bmatrix} O_J^J \tag{1.4}$$

$$V^* = -2O + \tfrac{5}{2} O_0^0 - \tfrac{1}{2} \sum_J O_J^J \tag{1.5}$$

whereby the J summations extend over the even-integer values from 0 to 4.

The conclusions that have been obtained (De Meyer and Vanden Berghe 1980) are the following.

(a) Operators commuting with N, J^2, V^* and the angular projection operator J_0, and being independent of these four operators, should at least contain terms built up with four phonon creation and four annihilation operators.

(b) They can be written in the following particular form:

$$\begin{aligned} X[\{f\}] = {} & f_{20}^{20} O_{20}^{20} + f_{02}^{02} O_{02}^{02} + f_{23}^{23} O_{23}^{23} + f_{24}^{24} O_{24}^{24} + f_{46}^{46} O_{46}^{46} + f_{020}^{020} O_{020}^{020} + f_{022}^{022} O_{022}^{022} \\ & + f_{202}^{202} O_{202}^{202} + f_{022}^{202} O_{022}^{202} + f_{202}^{022} O_{202}^{022} + f_{024}^{024} O_{024}^{024} + f_{234}^{234} O_{234}^{234} + f_{024}^{234} O_{024}^{234} \\ & + f_{234}^{024} O_{234}^{024} + f_{235}^{235} O_{235}^{235} + f_{246}^{246} O_{246}^{246} + f_{468}^{468} O_{468}^{468} \end{aligned} \tag{1.6}$$

where the real variables $f_{J_2 \ldots J_N}^{I_2 \ldots J_N}$ can be determined as follows:

$$\begin{aligned} & X[f_{20}^{20}, f_{02}^{02}, f_{23}^{23}, f_{24}^{24}, f_{46}^{46}; f_{020}^{020}, f_{022}^{022}, f_{202}^{202}, f_{022}^{202}, f_{202}^{022}, f_{024}^{024}, f_{234}^{234}, f_{024}^{234}, f_{234}^{024}, f_{235}^{235}, f_{246}^{246}, f_{468}^{468}] \\ & \quad = X\Big[\frac{25y - 11z}{14} - 2{\cdot}4053b - 0{\cdot}7143c, x, y, z, -\frac{55}{28}(y - z) \\ & \quad -6{\cdot}6147b - 1{\cdot}9643c; e, g, -3a + 2{\cdot}4053b \\ & \quad +0{\cdot}7143c, a, a, h, c, b, b, -2{\cdot}0952a + 2{\cdot}3926b \\ & \quad +0{\cdot}0816c + 0{\cdot}8658d, d, 1{\cdot}2768a - 2{\cdot}2180b - 0{\cdot}4559c + 0{\cdot}5804d\Big], \\ & \quad (x, y, z, a, b, c, d, e, g, h \in \mathrm{R}). \end{aligned} \tag{1.7}$$

One should, however, notice that such operators are only determined upon terms of the form N^2, J^2, V^* and N, with which they commute.

(c) Nine independent operators of the form (1.6) can be constructed out of the operators (1.3)–(1.5), i.e. N^3, J^2N, V^*N, N^4, J^2N^2, V^*N^2, J^4, V^{*2}, J^2V^*. In table 1 the values, which take the ten independent parameters x, y, z, a, b, c, d, e, g and h for each of these operators, are summarised.

Table 1. The operators N^3, J^2N, V^*N, N^4, J^4, V^{*2}, J^2N^2, V^*N^2 and J^2V^* expressed in canonical form.

	N^3	J^2N	V^*N	N^4	J^4	V^{*2}	J^2N^2	V^*N^2	J^2V^*
x	2·1429	−4·2857	1·4286	12·8571	−102·8557	−8·5715	−8·5712	2·8571	25·7145
y	1·4	−1·4	−0·7	8·4	−39·2	4·2	1·4	−6·3	1·4
z	1·9092	0·6367	−0·9546	11·4557	−38·1695	5·7277	14·6402	−8·5916	−8·2751
a	0	0	0	−0·9091	−14·5454	−0·2273	1·3637	0·4545	−0·6818
b	0	0	0	−0·7850	−22·7640	−0·1962	0·2617	0·3925	−0·1309
c	0	0	0	2·6434	20·6574	0·6608	−0·8811	−1·3217	0·4406
d	0	0	0	2·8	−11·2	0·7	4·2	−1·4	−2·1
e	0	0	0	2·1429	51·4286	4·2857	−4·2858	1·4286	−12·8571
g	0	0	0	3·6364	58·1816	−1·5909	−5·4546	0·6818	−4·2727
h	0	0	0	3·5665	−46·5738	−1·6084	−1·1888	0·7168	12·3776

(d) Any operator of the form (1.6) and (1.7), which is independent of the nine operators listed at the head of table 1, can be used as fifth label generating operator. The simplest one follows from the choice

$$d = 1 \qquad a = b = c = e = g = h = x = y = z = 0,$$

i.e.

$$S = 0{\cdot}8658\,O_{235}^{235} + O_{246}^{246} + 0{\cdot}5804\,O_{468}^{468}. \tag{1.8}$$

Weber *et al* (1966) have proposed some operators which could be used for the complete classification of the states considered. They are constructed in terms of the generators of the R(5) Lie algebra. In the first part of this paper a connection will be made between the operators (1.6)–(1.7) and the ones proposed by Weber *et al* (1966). It will also be shown that the second Casimir operator of the R(5) group is not linearly independent of the nine operators listed in table 1 and therefore not useful for the classification of the phonon states under consideration. Very recently Hughes and Yadegar (1978) have developed a method by which R(3) scalar operators can be constructed for low-dimensional groups possessing an R(3) subgroup. We shall state the form of this operator obtained by applying that method to the R(5) group and show its relation to the operators of the form (1.6)–(1.7) and to the ones proposed by Weber *et al* (1966).

In the second part of this paper a method will be presented by which it is possible to calculate the eigenvalues of the operator S (equation 1.8). Therefore the quadrupole-phonon states constructed by Vanden Berghe and De Meyer (1979b) will be used. The proposed method will be applied to derive the eigenvalues of the $N = 6$ states with total angular momentum $J_N = 6$, which are the first states degenerate with respect to the seniority.

2. Previously proposed fifth label generating operators

2.1. *Operators proposed by Weber et al* (1966)

Weber *et al* (1966) have proposed that one of the three following operators could eventually be used for the classification of the quadrupole-phonon states:

$$S_k = [((b^+b)^3(b^+b)^3)^{2k}((b^+b)^3(b^+b)^3)^{2k}]^0 \qquad k = 1, 2, 3. \tag{2.1}$$

These operator forms belong to a larger class of operators which we will study here, i.e.

$$Q(k_1k_2k_3k_4k_5) = \sqrt{2k_5+1}[((b^+b)^{k_1}(b^+b)^{k_2})^{k_5}((b^+b)^{k_3}(b^+b)^{k_4})^{k_5}]^0$$
$$k_1, k_2, k_3, k_4 = 1 \text{ or } 3. \tag{2.2}$$

Since these operators are all R(3) scalars, they commute with J^2 and J_0. Moreover, as the forms $(b^+b)^k_m$ ($k = 1$ or 3) are the generators of the R(5) group, they all commute with its Casimir operator V^*. Due to the particular form of (2.2) the commutator $[Q, N]$ is also zero. These Q operators fulfil the conditions, mentioned in the introduction, to be a fifth label generating operator. In this way it should be possible to transform (2.2) into the canonical form (1.6)–(1.7). This transformation is rather tedious and we have merely stated the result in appendix 1. The $12j$ symbols of the first and second kind and the $15j$ symbol, appearing in that result, are defined according to Yutsis *et al* (1962). By introducing the different possible values for the set

$(k_1, k_2, k_3, k_4, k_5)$ into (A1.1) it is now a matter of straightforward calculation to obtain the results for the ten independent parameters x, y, z, a, b, c, d, e, g and h as defined by (1.6)–(1.7). These numbers are summarised in table 2 for the 14 different existing operators of the form (2.2). The parameters of the other Q operators, not mentioned in table 2, are, except for a phase factor, either equal to one of the 14 summarised results or can be brought in a simpler form consisting of scalar R(3) operators constructed in terms of two R(5) generators. The Q operators with $k_1 = k_2$, $k_3 = k_4$ and k_5 odd belong to the last-quoted kind. Indeed

$$\sqrt{4k_5+3}[((b^+b)^{k_1}(b^+b)^{k_1})^{2k_5+1}((b^+b)^{k_3}(b^+b)^{k_3})^{2k_5+1}]^{(0)}$$

$$=\tfrac{1}{4}\sum_{M_5m_1m_2m_3m_4}(-1)^{1+M_5}\langle k_1m_1k_1m_2|2k_5+1M_5\rangle\langle k_3m_3k_3m_4|2k_5+1-M_5\rangle$$

$$\times[(b^+b)^{k_1}_{m_1},(b^+b)^{k_1}_{m_2}][(b^+b)^{k_3}_{m_3},(b^+b)^{k_3}_{m_4}]. \qquad (2.3)$$

The commutators in (2.3) are derived by Weber *et al* (1966) to be

$$[(b^+b)^{k_1}_{m_1},(b^+b)^{k_1}_{m_2}]$$

$$=2(2k_1+1)\sum_{L,M}(-1)^M\sqrt{4L+3}$$

$$\times\begin{Bmatrix}k_1 & k_1 & 2L+1\\ 2 & 2 & 2\end{Bmatrix}\begin{pmatrix}k_1 & k_1 & 2L+1\\ -m_1 & -m_2 & M\end{pmatrix}(b^+b)^{2L+1}. \qquad (2.4)$$

Introducing (2.4) into (2.3) and performing the Racah algebra, one gets

$$Q(k_1k_1k_2k_22k_5+1)$$

$$=\sqrt{4k_5+3}\begin{Bmatrix}k_1 & k_1 & 2k_5+1\\ 2 & 2 & 2\end{Bmatrix}\begin{Bmatrix}k_3 & k_3 & 2k_5+1\\ 2 & 2 & 2\end{Bmatrix}$$

$$\times(2k_1+1)(2k_3+1)[(b^+b)^{2k_5+1}(b^+b)^{2k_5+1}]^0 \qquad (2.5)$$

showing that operators of this kind can all be expressed in terms of N, J^2, N^2 and V^*.

Not all of the 14Q operators given in table 2 are linearly independent of the Casimir operators of the U(5), R(5) and R(3) groups. Indeed it is easy to show analytically that

$$Q(11110)=1/300J^4 \qquad (2.6)$$

$$Q(33330)=1/7(V^*+1/10J^2)^2 \qquad (2.7)$$

$$Q(33110)=-1/(10\sqrt{21})J^2(V^*+1/10J^2). \qquad (2.8)$$

Besides these three relations which can be simply understood, there exists one more. This supplementary one has been derived very recently (Vanden Berghe and De Meyer 1979a), i.e.

$$Q(11112)=1/300(\tfrac{2}{3}J^4-\tfrac{1}{2}J^2). \qquad (2.9)$$

We remark here that these four relations can be numerically checked by using tables 1 and 2.

Table 2. The operator $Q(k_1, k_2, k_3, k_4, k_5)$ as defined by (2.2) in canonical form.

k_1	k_2	k_3	k_4	k_5	a	b	c	d	e	g	h	x	y	z
1	1	1	1	0	−0·04848	−0·07588	0·06886	−0·03733	0·17143	0·19394	−0·15524	−0·34285	−0·13067	−0·12727
1	1	1	1	2	−0·09697	−0·15176	0·13771	−0·07467	0·34286	0·38788	−0·31049	−0·68570	−0·26134	−0·25454
3	3	3	3	0	−0·07272	−0·06429	0·13650	0·02400	0·31837	−0·28052	0·00639	−0·63672	0·58400	0·52727
3	3	3	3	2	0·16493	−0·06815	0·23015	0·08200	0·15918	−0·15974	−0·11918	−0·31836	0·10867	0·59242
3	3	3	3	4	−0·54758	−0·37700	0·14403	−0·04255	0·95510	−0·80968	0·21943	−1·91014	2·05199	1·04379
3	3	3	3	6	−0·05372	−0·09690	0·44450	0·10454	0·79591	−0·71369	−0·06189	−1·59179	1·34333	1·52739
3	3	1	1	0	0·04062	0·05253	−0·05469	0·07027	0·16834	−0·02281	−0·12956	−0·33668	0·05499	0·26384
3	3	1	1	2	0·05386	−0·02028	0·20548	−0·00998	0·04582	−0·01498	−0·04149	−0·09163	−0·01621	0·14456
3	3	3	1	2	0·13614	0·07899	0·12065	0·08733	0·13469	−0·15173	−0·09700	−0·26938	0·11733	0·51515
3	3	3	1	4	0·17416	0·10105	0·15434	0·11172	0·17230	−0·19409	−0·12409	−0·34459	0·15009	0·65898
1	1	3	1	2	0·10569	0·06084	0·13187	0·02993	−0·13745	−0·22231	0·12447	0·27490	0·12348	0·28062
3	1	3	1	2	0·07965	0·04647	0·05016	0·07133	0·20816	−0·03290	−0·16354	−0·41631	0·05133	0·36515
3	1	3	1	3	0·02083	0·09921	−0·27576	0·11667	0·21429	−0·02083	−0·15909	−0·42856	0·09917	0·26780
3	1	3	1	4	0·11315	0·09504	−0·02503	0·13400	0·34898	−0·05081	−0·27108	−0·69794	0·10150	0·57613

With the results obtained, the behaviour of the R(5) Casimir operator of order 4 can be investigated. Following Weber *et al* (1966) it can be defined as follows:

$$
\begin{aligned}
V_1 = 2^4 \sum_{\substack{k_1k_2k_3k_4\\ L_1L_2L_3L_4}} \sum_{\substack{m_1m_2m_3m_4\\ \lambda_1\lambda_2\lambda_3\lambda_4}} & (b^+b)^{k_1}_{m_1}(b^+b)^{k_2}_{m_2}(b^+b)^{k_3}_{m_3}(b^+b)^{k_4}_{m_4} \\
& (-1)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4}(2L_1+1)(2L_2+1)(2L_3+1)(2L_4+1) \\
& \times[(2k_1+1)(2k_2+1)(2k_3+1)(2k_4+1)]^{1/2} \\
& \times\begin{Bmatrix} k_1 & L_1 & L_2 \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} k_2 & L_2 & L_3 \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} k_3 & L_3 & L_4 \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} k_4 & L_4 & L_1 \\ 2 & 2 & 2 \end{Bmatrix} \\
& \times\begin{pmatrix} k_1 & L_1 & L_2 \\ -m_1 & -\lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} k_2 & L_2 & L_3 \\ -m_2 & -\lambda_2 & \lambda_3 \end{pmatrix}\begin{pmatrix} k_3 & L_3 & L_4 \\ -m_3 & -\lambda_3 & \lambda_4 \end{pmatrix} \\
& \times\begin{pmatrix} k_4 & L_4 & L_1 \\ -m_4 & -\lambda_4 & \lambda_1 \end{pmatrix}.
\end{aligned} \tag{2.10}
$$

Note that the k_i and L_i ($i=1, 2, 3, 4$) summation indices can only take the values 1 and 3. This biquadratic invariant can be transformed with the help of Racah algebra:

$$
\begin{aligned}
V_1 = 2^4 \sum_{\substack{k_1k_2k_3k_4k_5\\ L_1L_2L_3L_4}} & (-1)^{k_5}Q(k_1k_2k_3k_4k_5)(2L_1+1)(2L_2+1)(2L_3+1)(2L_4+1) \\
& \times[(2k_1+1)(2k_2+1)(2k_3+1)(2k_4+1)]^{1/2} \\
& \times\begin{Bmatrix} k_1 & L_1 & L_2 \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} k_2 & L_2 & L_3 \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} k_3 & L_3 & L_4 \\ 2 & 2 & 2 \end{Bmatrix} \\
& \times\begin{Bmatrix} k_4 & L_4 & L_1 \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} L_4 & L_3 & k_3 \\ k_5 & k_4 & L_1 \end{Bmatrix}\begin{Bmatrix} L_1 & L_2 & k_1 \\ k_2 & k_5 & L_3 \end{Bmatrix}.
\end{aligned} \tag{2.11}
$$

Introducing the canonical form of the Q operators (see appendix 1) and performing numerically the various summations, V_1 can be written as follows:

$$
\begin{aligned}
V_1 = & -0{\cdot}34091 O^{202}_{022} - 0{\cdot}29434 O^{234}_{024} + 0{\cdot}99121 O^{234}_{234} + 1{\cdot}05 O^{246}_{246} + 6{\cdot}42855 O^{020}_{020} \\
& - 2{\cdot}38628 O^{022}_{022} - 2{\cdot}41257 O^{024}_{024} - 12{\cdot}85671 O^{02}_{02} + 6{\cdot}3 O^{23}_{23} \\
& + 8{\cdot}59155 O^{24}_{24} - 7{\cdot}5 O^{0}_{0} + 13{\cdot}125(O^{2}_{2} + O^{4}_{4}) + 7{\cdot}5N.
\end{aligned} \tag{2.12}
$$

Comparing this expression with the canonical expansion of V^{*2} (see table 1) it is straightforward to deduce, within the accuracy of our numerical calculations, that:

$$V_1 = \tfrac{3}{2}V^{*2} - \tfrac{3}{4}V^* \tag{2.13}$$

showing that the quadratic and quartic Casimir operators are not independent. This result is obvious since we are dealing with symmetric representations of R(5), which need just one representation label. On the other hand, the explicit relationship between V_1 and V^* has also been obtained by a completely different method by Nwachuku and Rashid (1977) in their study of the eigenvalues of the Casimir operators of the orthogonal and symplectic groups for the special case of a completely symmetric representation.

2.2. The operator suggested by Hughes and Yadegar (1978)

Very recently Hughes and Yadegar (1978) have developed a general method by which R(3) shift and scalar operators can be constructed for low-dimensional groups possessing an R(3) subgroup. Hughes has shown that these operators, which he denotes as O_1^k, are useful to solve the eventual state labelling problems. This method can also be applied to the R(5) group. Although the construction of all O_1^k is outside the scope of the present work, it is very interesting to discuss the properties of the occurring scalar operator $O_1^{k=0}$.

This scalar operator for the R(5) case is defined as follows (Hughes and Yadegar 1978)

$$O_1^{k=0} = \gamma_0(l, m)Q_0 + \sum_{\mu=1}^{3} [\gamma_\mu^0(l, m)Q_{+\mu} - \gamma_\mu^0(l, -m)Q_{-\mu}] \tag{2.14}$$

where for $\mu = 0, 1, 2, 3$

$$\gamma_\mu^0(l, m) = (-1)^{3+3l-m}\left[\frac{(3!)^2(2l+4)!\,(l-m-\mu)!\,(l+m)!}{6!\,(2l-3)!\,(l-m)!\,(l+m+\mu)!}\right]^{1/2}\begin{pmatrix} 3 & l & l \\ \mu & -\mu-m & m \end{pmatrix} \tag{2.15}$$

and

$$Q_{\pm\mu} = T(3, \mp\mu)(J_\pm)^\mu \text{ for } \mu \geq 0. \tag{2.16}$$

In (2.16) $J_\pm$ are, together with J_0, the generators of the R(3) subgroup of R(5). They are defined in terms of the $(b^+b)_m^k$ introduced in (2.1) and (2.2) in the following way:

$$J_0 = \sqrt{10}(b^+b)_0^1 \qquad J_\pm = \mp 2\sqrt{5}(b^+b)_{\pm 1}^1. \tag{2.17}$$

The $T(3, \mu)$ ($\mu = -3, -2, \ldots, +3$) represent the seven-dimensional tensor representation of the other generators of R(5). They can be written in the following form:

$$T(3, \mu) = (b^+b)_\mu^3 \equiv q_\mu. \tag{2.18}$$

The generators (2.17) and (2.18) satisfy the commutation relations

$$\begin{aligned} &[J_0, J_\pm] = J_\pm \qquad &&[J_0, q_\mu] = \mu q_\mu \\ &[J_+, J_-] = 2J_0 \qquad &&[J_\pm, q_\mu] = [(3\mp\mu)(3\pm\mu+1)]^{1/2} q_{\mu\pm 1}. \end{aligned} \tag{2.19}$$

Introducing (2.17) and (2.18) into (2.14) and replacing the combination $l(l+1)$ by J^2 and m by J_0, it is easy to verify that

$$\begin{aligned} O_1^0 = &-\frac{2}{\sqrt{5}} q_0 J_0(3J^2 - 5J_0^2 - 1) - \sqrt{3/5}\, q_{-1}J_+[J^2 - 5J_0(J_0+1) - 2] + \sqrt{6}\, q_{-2}J_+^2(J_0+1) \\ &+ q_{-3}J_+^3 + \sqrt{3/5}\, q_{+1}J_-[J^2 - 5J_0(J_0-1) - 2] + \sqrt{6}\, q_{+2}J_-^2(J_0-1) - q_{+3}J_-^3. \end{aligned} \tag{2.20}$$

Since O_1^0 is a scalar operator, containing terms built up with four creation and annihilation operators, and since it commutes with the Casimir operators of the groups belonging to the chain U(5), R(5), R(3) and R(2), one can expect that it can be brought in the form (1.6). A straightforward calculation shows that

$$O_1^0 = -40\sqrt{7}\, Q(11312) \tag{2.21}$$

with Q defined by (2.2). By this relation and with the help of table 2 the reader can easily calculate the values of the parameters occurring in (1.7). Moreover, Hughes' method provides an algorithm for deriving analytically the eigenvalues of operators of the form O_1^0, thereby, since this operator is Hermitian, giving an orthogonal solution to the state labelling problem. We are currently investigating this way of working and hope to report on it in the near future.

3. Calculation of the eigenvalues of the operator S

Since the operator S (equation 1.8) is expressed in a second quantisation form it should be preferable to have the quadrupole-phonon wavefunctions available in terms of a coupling between phonon creation operators. Recently Vanden Berghe and De Meyer (1979b) have presented such a construction scheme. A wavefunction specified by the phonon number N, the seniority v, the number of zero coupled triplets μ, a total angular momentum J_N and its projection M_N can be denoted as follows:

$$|N, v, \mu, J_N, M_N\rangle = \sum a^{(l)}_{N,v,\mu,J_N} |\{J_i\}, i = 0, 1, \ldots, N\rangle \tag{3.1}$$

where the summation extends over the linear independent basis vectors and where the $a^{(l)}_{N,v,\mu,J_N}$ are the weights tabulated by Vanden Berghe and De Meyer (1979b). The state vectors $|\{J_i\}\rangle$ have the formal structure

$$|\{J_i\}, i = 0, 1, \ldots, N\rangle \equiv J_N[b_2^+ \otimes J_{N-1}[\ldots J_2][b_2^+ \otimes J_1][b_2^+ \otimes J_0]\ldots]|0\rangle \tag{3.2}$$

with $J_1 = 2$ since the choice $J_0 = 0$ has been made for reasons of symmetry. We remark that, because of notational convenience, the M_N quantum number has not been denoted. The wavefunctions (3.1) are orthonormalised and the eventual seniority degeneracy is solved by the quantum number μ, however, not related to the eigenvalue of an operator.

Applying S to a general state (3.1) is very laborious. Since the eigenvalues of S are independent of M_N no great loss of generality and a good deal of simplification results from considering states of maximum projection $M_N = J_N$. By doing this one can make use of the property that each of the basis vectors can be expressed in terms of products of elementary permissible diagrams (EPD) as introduced by Chacon *et al* (1976). Moreover, since $[S, V^*] = 0$ it follows that seniority non-degenerate states of the type (3.1) are also eigenstates of S. If, however, two seniority degenerate states of the form (3.1), i.e. $|N, v, \mu_1, J_N, M_N\rangle$ and $|N, v, \mu_2, J_N, M_N\rangle$ occur, it follows from the condition $[S, V^*] = 0$ that one can expect that $S|N, v, \mu_i, J_N, M_N\rangle$ ($i = 1$ or 2) should be a linear combination of both states.

Let us consider now as an example the $N = 6$ states with $J_N = 6$ and $M_N = 6$. There exist three such states, one $v = 4$ and two $v = 6$ states. They are written down as follows in the form (3.1) (Vanden Berghe and De Meyer 1979b)

$$|N = 6, v = 4, \mu = 0, J_N = 6, M_N = 6\rangle = 0{\cdot}14979|0, 2, 0, 2, 2, 4, 6\rangle_{M_N=6} \tag{3.3}$$

$$|N = 6, v = 6, \mu = 1, J_N = 6, M_N = 6\rangle$$
$$= -0{\cdot}07768|0, 2, 0, 2, 2, 4, 6\rangle_{M_N=6} + 0{\cdot}11220|0, 2, 2, 0, 2, 4, 6\rangle_{M_N=6} \tag{3.4}$$

$$|N = 6, v = 6, \mu = 0, J_N = 6, M_N = 6\rangle$$
$$= 0{\cdot}02377|0, 2, 0, 2, 2, 4, 6\rangle_{M_N=6} - 0{\cdot}01147|0, 2, 2, 0, 2, 4, 6\rangle_{M_N=6}$$
$$+ 0{\cdot}08692|0, 2, 2, 3, 4, 5, 6\rangle_{M_N=6}. \tag{3.5}$$

It is easy to show that the following relations yield between the occurring basis vectors and the combinations of the existing EPD (Chacon *et al* 1976)

$$\frac{1}{\sqrt{35}}(2,0)(1,2)^2(2,2)=|0,2,0,2,2,4,6\rangle_{M_N=6} \tag{3.6}$$

$$\frac{1}{\sqrt{35}}(3,0)(1,2)^3=|0,2,2,0,2,4,6\rangle_{M_N=6} \tag{3.7}$$

$$\frac{1}{7\sqrt{7}}(2,2)^3=\sum_{J_1J_2J_3J_4} 15\sqrt{5}[(2J_1+1)(2J_2+1)(2J_3+1)(2J_4+1)]^{1/2}$$

$$\times(-1)^{J_1+J_4}\begin{Bmatrix}2 & J_2 & 2\\ J_1 & 2 & 2\end{Bmatrix}\begin{Bmatrix}2 & J_4 & J_2\\ J_3 & 2 & 2\end{Bmatrix}\begin{Bmatrix}2 & 2 & 2\\ J_2 & J_4 & 2\\ 2 & 6 & 4\end{Bmatrix}$$

$$\times|0,2,J_1,J_2,J_3,J_4,6\rangle_{M_N=6}$$

$$=1\cdot14997|0,2,0,2,2,4,6\rangle_{M_N=6}-0\cdot51109|0,2,2,0,2,4,6\rangle_{M_N=6}$$

$$+1\cdot13388|0,2,2,3,4,5,6\rangle_{M_N=6}. \tag{3.8}$$

In order to obtain the result (3.8) we have made use of the fact that all $|0,2,J_1,J_2,J_3,J_4,6\rangle$ states can be developed in terms of the three chosen basis vectors (Vanden Berghe and De Meyer 1979b). Due to the relations (3.6)–(3.8) the three $N=6$ states (3.3)–(3.5) can be expressed in terms of the considered EPD. It is now a matter of straightforward application of Racah algebra to derive the results for $S(2,0)(1,2)^2(2,2)$, $S(3,0)(1,2)^3$ and $S(2,2)^3$. As an example we present in appendix 2 the result of the application of an operator of the form $O^{J_1J_2J_3}_{J_1'J_2'J_3'}$ on $(2,2)^3$. By using this expression and the other analogous ones, each of the present kets has to be developed in terms of the basis vectors. Since such operations necessarily require the knowledge of all existing relationships between the state vectors $|\{J_i\}\rangle$, and therefore are quite involved, we have invoked computer assistance for numerical treatment. The numerical results are summarised in table 3. These results can now be used for the application of S on the $N=6$ states (3.3)–(3.6). The following results are then obtained:

$$S|6,4,0,6,6\rangle=50\cdot6502|6,4,0,6,6\rangle$$

$$S|6,6,1,6,6\rangle$$
$$=-6\cdot1049|0,2,0,2,2,4,6\rangle+8\cdot8588|0,2,2,0,2,4,6\rangle$$
$$+0\cdot1552|0,2,2,3,4,5,6\rangle$$
$$=79\cdot1379|6,6,1,6,6\rangle+1\cdot7851|6,6,0,6,6\rangle$$

$$S|6,6,0,6,6\rangle$$
$$=1\cdot0571|0,2,0,2,2,4,6\rangle-0\cdot3767|0,2,2,0,2,4,6\rangle$$
$$+4\cdot3730|0,2,2,3,4,5,6\rangle$$
$$=1\cdot7858|6,6,1,6,6\rangle+50\cdot3111|6,6,0,6,6\rangle.$$

These results clearly indicate that the wavefunction of the first considered 6^+ state, which is not degenerate with respect to seniority, is indeed an eigenvector of the S

Table 3. The numerical results of the operation of the operator $S(1.8)$ on $(1/\sqrt{35})$ $(2,0)(1,2)^2(2,2)$, $(1/\sqrt{35})(3,0)(1,2)^3$ and $(1/7\sqrt{7})(2,2)^3$.

State	Operator	Basis vectors $\|0,2,0,2,2,4,6\rangle$	$\|0,2,2,0,2,4,6\rangle$	$\|0,2,2,3,4,5,6\rangle$
$\frac{1}{\sqrt{35}}(2,0)(1,2)^2(2,2)$	O_{246}^{246}	29·2673	6·1091	4·4255
	$0{\cdot}8658\,O_{235}^{235}$	8·1040	−8·1040	−2·1074
	$0{\cdot}5804\,O_{468}^{468}$	13·2789	1·9949	−2·3183
	Total	50·6502	0·0000	−0·0002
$\frac{1}{\sqrt{35}}(3,0)(1,2)^3$	O_{246}^{246}	17·3920	24·2181	−2·2128
	$0{\cdot}8658\,O_{235}^{235}$	−40·2080	40·2080	3·1610
	$0{\cdot}5804\,O_{468}^{468}$	3·4716	14·5297	0·4347
	Total	−19·3444	78·9558	1·3829
$\frac{1}{7\sqrt{7}}(2,2)^3$	O_{246}^{246}	46·5962	−16·7269	28·2734
	$0{\cdot}8658\,O_{235}^{235}$	16·7269	−16·7269	28·2735
	$0{\cdot}5804\,O_{468}^{468}$	0·0	0·0	0·0
	Total	63·3231	−33·4538	56·5469

operator. We remark, however, that in the intermediate results (table 3) contributions to all basis vectors are obtained. The wavefunctions of the other 6^+ states, which are degenerate with respect to seniority, are no eigenvectors of the S operator. Due to the fact that S is a Hermitian operator, one expects that

$$\langle 6,6,0,6,6|S|6,6,1,6,6\rangle = \langle 6,6,1,6,6|S|6,6,0,6,6\rangle$$

a condition, which within the accuracy of our numerical calculations, is fulfilled.

It is now a matter of straightforward calculation to determine the $N=6$, $J_N=6^+$ orthonormalised wavefunctions which are simultaneously eigenstates of N, J^2, J_0, V^* and S. One finds:

$$|6_1^+\rangle = 0{\cdot}1498|0,2,0,2,2,4,6\rangle \tag{3.9}$$

$$\begin{aligned} |6_2^+\rangle &= 0{\cdot}9981|6,6,1,6,6\rangle + 0{\cdot}0616|6,6,0,6,6\rangle \\ &= -0{\cdot}0761|0,2,0,2,2,4,6\rangle + 0{\cdot}1113|0,2,2,0,2,4,6\rangle \\ &\quad + 0{\cdot}0054|0,2,2,3,4,5,6\rangle \end{aligned} \tag{3.10}$$

$$\begin{aligned} |6_3^+\rangle &= -0{\cdot}0616|6,6,1,6,6\rangle + 0{\cdot}9981|6,6,0,6,6\rangle \\ &= 0{\cdot}0285|0,2,0,2,2,4,6\rangle - 0{\cdot}0184|0,2,2,0,2,4,6\rangle \\ &\quad + 0{\cdot}0868|0,2,2,3,4,5,6\rangle. \end{aligned} \tag{3.11}$$

The eigenvalues of S associated with these states are, respectively, 50·6502, 79·2477 and 50·2009. This kind of calculation can be performed for each seniority degenerate state. Although the numerical manipulations are quite involved the obtained eigenvalues clearly distinguish the several states.

4. Conclusions

In the first part of this paper the properties of a fifth label generating operator for quadrupole phonon states have been reviewed. It has been shown that the R(3) scalar operators of biquadratic form in the R(5) generators as proposed by Weber *et al* (1966) can fulfil these specific properties. Moreover, it has been proved that there are only ten such biquadratic forms, which are linearly independent of the U(5), R(5), R(3) and R(2) Casimir operators. A link was made between the operators constructed with the help of pure algebraical methods and the operators following from pure group theoretical principles. At the same time it has been confirmed that the R(5) Casimir operator of order 4 can be completely expressed in terms of the second-order R(5) Casimir operator. We also have shown that the R(3) scalar operator, constructed by using a method recently proposed by Hughes and Yadegar (1978), was equal, except for an overall numerical factor to one of the biquadratic forms just mentioned.

In the second part an algorithm is given for the calculation of the eigenvalues of the Hermitian operator S introduced by De Meyer and Vanden Berghe (1980). This method is worked out in detail for the $N=6$ states with angular momentum 6. It is shown that the eigenvalues obtained clearly distinguish the three $I^\pi=6^+$ states. Unfortunately we were not able to interpret the physical content of such eigenvalue. Since, however, an infinity of such fifth label generating operators exists, it is quite difficult to believe that an operator S selected at random should be related to a physical property.

Appendix 1. The operator $Q(k_1k_2k_3k_4k_5)$ in canonical form

$$
\begin{aligned}
Q(k_1k_2k_3k_4k_5)
&= (2k_5+1)^{1/2}[((b^+b)^{k_1}(b^+b)^{k_2})^{k_5}((b^+b)^{k_3}(b^+b)^{k_4})^{k_5}]^{(0)}\\
&= [(2k_1+1)(2k_2+1)(2k_3+1)(2k_4+1)]^{1/2}(2k_5+1)\\
&\quad\times\Bigg\{\begin{Bmatrix}2 & 2 & k_5\\ k_1 & k_2 & 2\end{Bmatrix}\begin{Bmatrix}2 & 2 & k_5\\ k_3 & k_4 & 2\end{Bmatrix}\Bigg[\frac{(-1)^{k_5}}{5}O\\
&\quad+\sum_{J_2} O^{J_2}_{J_2}\bigg((-1)^{k_5}\begin{Bmatrix}2 & 2 & J_2\\ 2 & 2 & k_4\end{Bmatrix}+(-1)^{k_5}\begin{Bmatrix}2 & 2 & J_2\\ 2 & 2 & k_5\end{Bmatrix}\\
&\quad+(-1)^{k_5}\begin{Bmatrix}2 & 2 & J_2\\ 2 & 2 & k_1\end{Bmatrix}+(-1)^{k_3+k_4}\begin{Bmatrix}2 & 2 & J_2\\ 2 & 2 & k_3\end{Bmatrix}\\
&\quad+(-1)^{k_1+k_2}\begin{Bmatrix}2 & 2 & J_2\\ 2 & 2 & k_2\end{Bmatrix}\bigg)\Bigg]+\sum_{J_2} O^{J_2}_{J_2}\left((-1)^{k_2+k_3+k_5}\begin{bmatrix}k_1 & 2 & k_4 & 2\\ 2 & k_5 & J_2 & 2\\ 2 & 2 & k_2 & k_3\end{bmatrix}\right.\\
&\quad\left.+(-1)^{k_2+k_3}\begin{bmatrix}k_2 & 2 & k_4 & 2\\ 2 & k_5 & J_2 & 2\\ 2 & 2 & k_1 & k_3\end{bmatrix}\right)\\
&\quad+\sum_{J_2J_3J_4}[(2J_2+1)(2J_3+1)]^{1/2}O^{J_3J_4}_{J_2J_4}
\end{aligned}
$$

$$
\times\Bigg((-1)^{k_4+k_5+J_3+J_4}\begin{Bmatrix} 2 & 2 & k_5 \\ k_1 & k_2 & 2 \end{Bmatrix}\begin{Bmatrix} 2 & 2 & k_4 \\ J_2 & J_3 & J_4 \end{Bmatrix}\begin{Bmatrix} k_5 & k_3 & k_4 \\ 2 & 2 & J_2 \\ 2 & 2 & J_3 \end{Bmatrix}
$$

$$
+(-1)^{J_3+J_4}\begin{Bmatrix} 2 & 2 & k_5 \\ k_3 & k_4 & 2 \end{Bmatrix}\begin{Bmatrix} 2 & 2 & k_5 \\ J_2 & J_3 & J_4 \end{Bmatrix}\begin{Bmatrix} k_1 & k_2 & k_5 \\ 2 & 2 & J_2 \\ 2 & 2 & J_3 \end{Bmatrix}
$$

$$
+(-1)^{k_1+k_2+k_3+J_2+J_4}\begin{Bmatrix} 2 & 2 & k_4 \\ J_2 & J_3 & J_4 \end{Bmatrix}\begin{Bmatrix} 2 & & 2 & & 2 & & J_2 & \\ & k_3 & & k_2 & & 2 & & J_3 \\ k_4 & & k_5 & & k_1 & & 2 & \end{Bmatrix}
$$

$$
+(-1)^{k_3+k_5+J_4}\begin{Bmatrix} 2 & 2 & k_4 \\ J_2 & J_3 & J_4 \end{Bmatrix}\begin{Bmatrix} 2 & & 2 & & 2 & & J_2 & \\ & k_3 & & k_1 & & 2 & & J_3 \\ k_4 & & k_5 & & k_2 & & 2 & \end{Bmatrix}
$$

$$
+(-1)^{k_1+k_2+J_4}\begin{Bmatrix} 2 & & 2 & & J_3 & & 2 & & k_1 & \\ & k_4 & & 2 & & J_4 & & 2 & & k_2 \\ k_5 & & k_3 & & 2 & & J_2 & & 2 & \end{Bmatrix}
$$

$$
+(-1)^{J_2+J_4}\begin{Bmatrix} 2 & & 2 & & J_3 & & 2 & & k_1 & \\ & k_4 & & 2 & & J_4 & & 2 & & k_2 \\ k_5 & & k_3 & & 2 & & J_2 & & 2 & \end{Bmatrix}\Bigg)
$$

$$
+\sum_{J_2J_3J_4J_5J_6}(-1)^{J_5+J_6+k_4}[(2J_2+1)(2J_3+1)(2J_4+1)(2J_5+1)]^{1/2}O^{J_3J_5J_6}_{J_2J_4J_6}
$$

$$
\times\begin{Bmatrix} 2 & J_6 & J_5 \\ J_4 & k_4 & 2 \end{Bmatrix}\begin{Bmatrix} k_1 & k_2 & k_5 \\ 2 & 2 & J_2 \\ 2 & 2 & J_3 \end{Bmatrix}\begin{Bmatrix} 2 & J_2 & J_4 \\ k_3 & k_5 & k_4 \\ 2 & J_3 & J_5 \end{Bmatrix}\Bigg\} \tag{A1.1}
$$

Appendix 2. The result for $O^{J_1J_2J_3}_{J_1J_2J_3}(2,2)^3$

$$
O^{J_1J_2J_3}_{J_1J_2J_3}\frac{(2,2)^3}{7\sqrt{7}}
$$

$$
=720\delta_{J_1,2}\sum_J[(2J+1)(2J_2+1)(2J_3+1)]^{1/2}(-1)^J
$$

$$
\times\begin{Bmatrix} J_2 & J & 2 \\ 2 & 2 & J_3 \end{Bmatrix}\begin{Bmatrix} 2 & J_2 & 2 \\ 2 & J & 6 \\ 2 & 2 & 4 \end{Bmatrix}|0,2,J_1,J_2,J_3,J,6\rangle_{M_N=6}
$$

$$
+1540\sqrt{5}\sum_J[(2J+1)(2J_1+1)(2J_2+1)(2J_3+1)]^{1/2}(-1)^{J_1+J_2}
$$

$$
\times\begin{Bmatrix} J_2 & J & 2 \\ 2 & 2 & J_3 \end{Bmatrix}\begin{Bmatrix} 2 & 2 & J_2 \\ 2 & J_1 & 2 \end{Bmatrix}\begin{Bmatrix} J_2 & J & 2 \\ 2 & 2 & 2 \\ 2 & 6 & 4 \end{Bmatrix}|0,2,J_1,J_2,J_3,J,6\rangle_{M_N=6}
$$

$$+1540\sqrt{5}\sum_{JJ''}[(2J_1+1)(2J_2+1)(2J_3+1)(2J+1)]^{1/2}$$

$$\times(2J''+1)(-1)^{J_1+J_2+J+J''}\begin{Bmatrix}2 & J & J''\\ J_3 & 2 & 2\end{Bmatrix}$$

$$\times\begin{Bmatrix}J_1 & 2 & 2\\ J_2 & 2 & J_3\\ 2 & 2 & J''\end{Bmatrix}\begin{Bmatrix}2 & 2 & 2\\ J'' & J & 2\\ 2 & 6 & 4\end{Bmatrix}|0,2,J_1,J_2,J_3,J,6\rangle_{M_N=6}$$

$$+720\sqrt{5}\sum_{J}[(2J+1)(2J_1+1)(2J_2+1)(2J_3+1)]^{1/2}$$

$$\times\begin{Bmatrix}J_1 & J_3 & 2\\ 2 & 2 & J_2\end{Bmatrix}\begin{Bmatrix}4 & 6 & 2\\ 2 & 2 & J\end{Bmatrix}\begin{Bmatrix}J_1 & J_3 & 2\\ 2 & 2 & 2\\ 2 & J & 4\end{Bmatrix}|0,2,J_1,J_2,J_3,J,6\rangle_{M_N=6}.$$

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